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From (5) and (6), since $p=-\frac{3}{2}\sqrt{6}$ and $q=0$, $u=\sin^{-1}(-1)=-90°$, $v=\mathrm{Sinh}^{-1}(\frac{5}{2}\sqrt{2})=1.97544$, $\mathrm{Sinh}\frac{v}{3}=0.70709$, $\mathrm{Cosh}\frac{v}{3}=1.22473$;

$$x_1=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times1.22473+i\frac{\sqrt{3}}{2}\times0.70709)=-0.00001+i\times0.99997,$$

$$x_2=\frac{2(1-i)}{\sqrt{6}}(1.22473)=0.99998(1-i),$$

$$x_3=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times1.22473-i\frac{\sqrt{3}}{2}\times0.70709)=-0.99997+i\times0.00001.$$

These are from five-figure tables; the exact roots are i, $1-i$, and -1.

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ON THE EXPANSION OF DEVERTEBRATED THREE DIMENSIONAL DETERMINANTS AND THE EXTENSION OF CAYLEY'S EXPANSION THEOREM.

By ORLANDO S. STETSON, Syracuse University.

The primary object of this paper is to extend to three dimensional or cubical determinants* the general law for the expansion of a two dimensional devertebrated determinant† in terms of the elements of the principal diagonal of the given determinant and their co-axial minors or, in other words, to develop a general law for the expansion of a devertebrated cubical determinant in terms of the elements of its principal diagonal plane and their co-planar minors.

The secondary object is to show how Cayley's Expansion Theorem may be extended to cubical determinants with but slight modifications. A formula for the number of terms of a cubical determinant which are independent of the elements of the principal diagonal plane is also given.

Let $\triangle$ denote a cubical determinant of order n and let $\triangle'_a$ denote a new cubical determinant formed by adding a different variable to each of the n^2 elements of the principal diagonal plane.

Expanding $\triangle'_a$ as a cubical determinent with binomial elements, the term independent of the variable will be the given cubical determinant $\triangle$. The next n^2 terms are the products of the respective variables and their corresponding cubical minors of order $n-1$; similarly, the next $\frac{n^2(n-1)^2}{2!}$ terms are products

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*An exposition and bibliography of cubical determinants is given by E. R. Hedrick, *Annals of Mathematics*, Ser. 2, Vol. 1 (1900), pp. 49—67.

†Stetson, MONTHLY, Vol. XI (1904), pp. 166—168.

of pairs of variables of the principal diagonal plane and their corresponding cubical minors of order $n-2$, no two of the variables being common to the same plane, and so on.

Now, let us assume any a elements of the principal diagonal plane of the given cubical determinant to be equal to zero and let us replace each of the corresponding variables by the negative of the element, placing the remaining n^2-a variables equal to zero; the general law may then be stated as follows:

A devertebrated cubical determinant of order n containing a zero elements in its principal diagonal plane may be developed into a series of terms of which the first is the given cubical determinant, $\triangle$; the next a being the products of each of the a elements into its co-planar cubical minor of order $n-1$; the next set consisting of the products of every possible pair of the a elements of the principal diagonal plane and its coplanar cubical minor of order $n-2$, no two of the elements being common to a same plane, and so on; the signs of the respective terms are positive or negative according as the combinations of the a elements are even or odd.

By a proper selection of the a elements this general law may be seen to contain as a particular case the law for the expansion of a devertebrated cubical determinant in which the a zero elements lie along the principal diagonal of the ordinary determinant of the principal diagonal plane.

For $a=n^2$, the expansion will be seen to contain, also, the law for the expression of an invertebrate coplanar minor in terms of the vertebrate coplanar minors and the elements of the principal diagonal plane.

As illustrations, we may assume a cubical determinant, $\triangle'_a$ of order 3 in which the elements in parentheses designate the variables which are added to the elements of the principal diagonal plane. We will write the sheets under one another, thus—

$$\triangle'_a = \left\{ \begin{array}{l} \left\{ \begin{array}{lll} 111+(111) & 112 & 113 \\ 121 & 122+(122) & 123 \\ 131 & 132 & 133+(133) \end{array} \right. \\ \\ \qquad\quad \left\{ \begin{array}{lll} 211+(211) & 212 & 213 \\ 221 & 222+(222) & 223 \\ 231 & 232 & 233+(233) \end{array} \right. \\ \\ \qquad\qquad\qquad \left\{ \begin{array}{lll} 311+(311) & 312 & 313 \\ 321 & 322+(322) & 323 \\ 331 & 332 & 333+(333) \end{array} \right. \end{array} \right.$$

Expanding $\triangle'_a$ as a cubical determinant with binomial elements, the expansion may be written in the following form—

$$\begin{aligned} \triangle'_a = \triangle & +(111)[111]+(122)[122]+(133)[133]+(211)[211]+(222)[222] \\ & +(233)[233]+(311)[311]+(322)[322]+(333)[333] \\ & +(111)(222)[111,\ 222]+(111)(322)[111,\ 322]+(211)(122) \end{aligned}$$

$$
\begin{aligned}
&[211,\ 122]+(211)(322)[211,\ 322]+(311)(122)[311,\ 122]\\
&+(311)(222)[311,\ 222]+(122)(233)[122,\ 233]+(122)(333)\\
&[122,\ 333]+(222)(333)[222,\ 333]+(222)(133)[222,\ 133]\\
&+(322)(133)[322,\ 133]+(322)(233)[322,\ 233]+(133)(211)\\
&[133,\ 211]+(133)(311)[133,\ 311]+(233)(111)[233,\ 111]\\
&+(233)(311)[233,\ 311]+(333)(111)[333,\ 111]+(333)(211)\\
&[333,\ 211]+(111)(222)(333)+(111)(322)(233)\\
&+(122)(211)(333)+(122)(233)(311)+(133)(211)(322)\\
&+(133)(222)(311).
\end{aligned}
$$

For $a=1$, assume $(111)=-111$, and place the remaining variables equal to zero, then

$$\Delta'_1=\Delta-111[111].$$

For $a=2$, assume $(111)=-111$, $(333)=-333$ and place the remaining variables equal to zero, then

$$\Delta'_2=\Delta-111[111]-333[333]+111.222.333.$$

For $a=3$, assume $(111)=-111$, $(211)=-(222)=-222$, and place the remaining variables equal to zero, then

$$\Delta'_3=\Delta-111[111]-211[211]-222[222]+111.222.333.$$

We may now obtain very readily a formula for the number of terms in a cubical determinant which are independent of the elements of the principal diagonal plane.

In the preceding article it has been shown that the expansion of a devertebrated determinant may be put in the following form—

$$
\begin{aligned}
\Delta_a = \Delta &+ \Sigma(111)[111]+\Sigma(111)(222)[111,\ 222]\\
&+\Sigma(111)(222)(333)[111,\ 222,\ 333]+\ldots\ldots\\
&+\Sigma(111)(222)(333)\ldots\ldots aaa[111,\ 222,\ \ldots\ldots,\ aaa]\\
&+(111)(222)\ldots\ldots(nnn),
\end{aligned}
$$

no two elements of a product belonging to a same plane. Placing $a=n$, thereby making Δ_n an invertebrate cubical determinant, we have

$$
\begin{aligned}
\Delta_n=\Delta &-\Sigma\,111[111]+\Sigma\,111.222[111,\ 222]-\Sigma\,111.222.333[111,\ 222,\ 333]\\
&+\ldots\ldots+(-1)^r\,\Sigma\,111.222\ldots\ldots rrr[111,\ 222\ldots\ldots rrr]\\
&+\ldots\ldots(-1)^n 111.222.333\ldots\ldots nnn.
\end{aligned}
$$

Since there are $(n!)^2$ terms in the expansion of a cubical determinant of order n, the number of terms in the preceding invertebrate cubical determinant will be

$$\phi(n)=(n\,!)^2-n^2[(n-1)\,!]^2+\frac{n^2\,(n-1)^2}{1.2}[(n-2)\,!]+\dots\dots$$

$$-\frac{n^2(n-1)^2(n-2)^2}{1.2.3}[(n-3)\,!]+\dots\dots$$

$$+(-1)^r\frac{n^2(n-1)^2\dots\dots(n-r+1)^2}{1.2.3\dots\dots r}[(n-r)\,!]+\dots\dots$$

$$+(-1)^n\frac{n^2(n-1)^2\dots\dots3.2.1}{1.2.3\dots\dots n}.$$

Removing from each of the terms the common factor $[n\,!]^2$ and noticing that the first two terms are equal but of opposite sign, we have

$$\phi(n)=(n\,!)^2\left\{\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\dots\dots(-1)^n\frac{1}{1.2.3\dots\dots n}\right\}.$$

CAYLEY'S EXPANSION THEOREM.

Any cubical determinant may be developed into a series of terms of which the first is obtained by changing into zero all the elements of the principal diagonal plane, the next n^2 by multiplying each of the elements of the principal diagonal plane by its minor in the cubical determinant and altering the principal diagonal plane of the minor as the given cubical determinant was altered; the next $\frac{n^2\,(n-1)^2}{2\,!}$ by multiplying each pair of elements of the principal diagonal plane, no two coming from a same set of planes, by its minor in the cubical determinant and altering the principal diagonal plane of the minor as the original cubical determinant was altered, etc., the last $n\,!$ terms being every possible product, n at a time, of the elements of the principal diagonal plane, no two being taken from a same set.

By changing into zero all the elements of the principal diagonal plane we delete exactly those terms of the cubical determinant which contain any of these elements. The product of an element of the principal diagonal plane and its minor in the given cubical determinant gives exactly those terms of the given cubical determinant which contain that element.

Changing into zero all the elements of the principal diagonal plane of the minor we obtain the sum of all the terms of the cubical determinants which contain that element and *only* that element.

The expansion gives, therefore, first, all the terms of the determinant involving no element of the principal diagonal plane of the given cubical determinant; secondly, all those involving *only* one element; thirdly, all those involving *only* two elements, etc.

The number of terms in the expansion which contain only one element will be the same as the number of elements in the principal diagonal plane, or n^2.

Since in forming the pair of elements one and only one element of a pair can be taken from the same set of planes, it follows that $n(n-1)$ pairs can be taken from a same two sets of planes in which a corresponding pair of planes in the set are kept fixed and, therefore, in every possible way there can be formed

$$n(n-1)n_2 \text{ or } \frac{n^2\ (n-1)^2}{2\,!}$$

by choosing in every possible manner the pair of planes to be kept fixed.

It should be noticed that the expansion passes directly from those terms involving only $n-2$ elements of the principal diagonal plane to those involving them all. There would be no difficulty in stating at once the general law for the expansion of a cubical determinant in terms of the elements of the principal diagonal of the principal diagonal plane of the cubical determinant and their coaxial cubical minors.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

256. Proposed by **THEODORE L. DeLAND, Treasury Department, Washington, D. C.**

Three men, A, B, and C, rented a pasture for a fixed amount, each to pay per month in proportion to the stock pastured. During the first month A put in 3 horses and B and C each some horses, and B paid for the month $6, but A and C each defaulted payment. During the next month each put in one more horse, and C paid for the month $7.20, but A and B each defaulted payment. During the next month each put in one more horse, and A paid his bill for the month, $5, but B and C each defaulted.

Required: (1) the rent of the pasture per month; (2) the number of horses B and C each put in during the first month; and (3) the amount A, B, and C, each, owed for the unpaid service.

I. Solution by **S. A. COREY, Hiteman, Iowa.**

Let a, b, c=rate per horse per month for the first, second, and third month, respectively. Let x, y=number of horses put in the first month by B, and C, respectively. Let $n=3+x+y$=total number of horses in pasture first month. Let m=fixed monthly rental of pasture.

Then as A paid $5 for the third month's rental when he had 5 horses in the pasture, c=$1. As fewer horses were in the pasture the two preceding months the rate per horse per month was more than $1 for the first and second months,